AN UNSOLVABLE HYPOELLIPTIC DIFFERENTIAL OPERATOR

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ABSTRACT

It is proved that the differential operator $D_1 + ix_1 D_2^2$ is hypoelliptic everywhere, but is not locally solvable in any open set which intersects the line $x_1 = 0$. Thus, this operator is not contained in the usual classes of hypoelliptic differential operators. The proofs involve certain properties of the characteristic Cauchy problem for the backward heat operator.

1. Introduction

A differential operator $A(x, D)$ with C^∞ coefficients is called hypoelliptic if the equation $A(x,D)u = f$ with $f \in C^\infty$ has only C^∞ solutions. All the well known classes of hypoelliptic differential operators [2, 3, 4, 7] contain, together with every operator, also its formal adjoint. Since hypoellipticity implies certain a priori estimates (see e.g. [4]) and since a priori estimates for the formal adjoint of a differential operator imply local existence for the operator, it follows that the well known hypoelliptic operators are locally solvable. In this note we shall exhibit a hypoelliptic second order differential operator in two variables for which there exists a line such that the operator is not solvable at any point on this line. In particular, this operator is not contained in any of the classes considered in [2, 3, 4, 7] (in an open set which intersects the line) and its adjoint is not hypoelliptic there.

The differential operator is

$$A = D_1 + ix_1 D_2^2 \tag{1}$$

where, as usual, (x_1, x_2) denotes a point in the plane and $D_j = -i(\partial/\partial x_j)$, $j = 1, 2$, is the standard differentiation operator.

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The nature of A becomes more transparent if the coordinates $x_1^2/2 = t$, $x_2 = x$ are used. Then

$$A = -i\sqrt{2t}\left(\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}\right) \tag{2}$$

Note that on each of the half planes $\{x_1 > 0\}$ and $\{x_1 < 0\}$, A is equal to a C^∞ function times the backward heat operator (on each half plane separately). Thus, A is certainly hypoelliptic in the complement of the line $x_1 = 0$. In Section 2 we shall prove that A is hypoelliptic everywhere, and in Section 3 we shall prove that A is not solvable at any point on the line $x_1 = 0$. We recall that a differential operator P is called solvable at a point w if there is an open neighborhood V of w such that for every function $f \in C_0^\infty(V)$ the equation $Pu = f$ possesses a solution u in $\mathscr{D}'(V)$.

We remark that both the hypoellipticity and the unsolvability of A on $x_1 = 0$ are connected with the fact that the characteristic Cauchy problem is not well-posed for the backward heat operator. Thus, the behavior of a solution of $Au = f$ on the line $x_1 = 0$ is determined by the values of u elsewhere (note that $x_1 = 0$ corresponds to a time "later" than the time to which $x_1 \neq 0$ corresponds, whether x_1 is positive or negative, since $t = x_1^2/2$), which contributes to the smoothness of u near this line. The unsolvability of A near the line is also closely related to properties of the backward Cauchy problem, as will be elucidated in Section 3.

Note that our operator is a suitable modification of the operator $D_1 + ix_1D_2$, consider in [6]. Note also that the commutator $AA^* - A^*A = -2iD_2^2$ is of even order and its symbol does not change its sign, in contradistinction to the commutators of the operators considered in [2] and in [6].

2. Proof of the hypoellipticity of A:

Let u be a solution of the equation

$$Au = f \tag{3}$$

where $u \in \mathscr{D}'(\Omega)$, $f \in C^\infty(\Omega)$ and Ω is an open subset of the plane. As was shown briefly in the introduction, it is well known that $u \in C^\infty(\Omega \cap \{(x_1, x_2) : x_1 \neq 0\})$. Hence, we have only to show that u is infinitely differentiable also in the neighborhood of the line $x_1 = 0$. We may assume, therefore, that Ω is an open disk whose center lies on the line $x_1 = 0$, and since A is invariant under translations in the direction of the x_2 variable we may also assume that the center of Ω is at the origin. It suffices to prove that u is infinitely differentiable in a neighborhood V of the origin, where cl(V) (the closure of V) is contained in Ω. Let $\phi \in C_0^\infty(\Omega)$ be identically equal to 1 in a neighborhood V_1 of cl(V). Then

$$
\begin{aligned}
A\phi u &= \phi A u + (D_1\phi + ix_1 D_2^2\phi)u + 2ix_1(D_2\phi)(D_2 u) \\
&= g + (A\phi)u + 2ix_1(D_2\phi)(D_2 u)
\end{aligned} \tag{4}
$$

(where we set $\phi f = g$). Let us first introduce new coordinates $-(x_1^2/2) = t$, $x_2 = x$ for the open half plane $\{(x_1, x_2) : x_1 > 0\}$. Then

$$
\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\phi u = \frac{-ig}{\sqrt{-2t}} - \frac{i}{\sqrt{-2t}}(A\phi)u + 2(D_2\phi)(D_2 u) \tag{5}
$$

The function $\phi u(\sqrt{-2t}, x)$ vanishes identically on the line $t = -K$ if K is sufficiently large and is bounded in the set $\{(t,x) : -\infty < x < \infty, -K \leqq t \leqq -\varepsilon\}$ for every $\varepsilon > 0$. Hence, we may use the usual fundamental solution of the heat equation (5) and conclude that

$$
\begin{aligned}
(\phi u)\left(\sqrt{-2t}, x\right) &= \frac{H(t)}{\sqrt{4\pi t}} e^{-x^2/4t} * \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\phi u \\
&= \int_{-\infty}^{t} \frac{1}{\sqrt{4\pi(t-\tau)}} \int e^{-(x-y)^2/4(t-\tau)} \left[\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\phi u\right](\sqrt{-2\tau}, y)\, dy\, d\tau
\end{aligned} \tag{6}
$$

(here $H(t)$ denotes the Heaviside function) for negative values of t. Using (5), we see, more explicitly, that

$$
\begin{aligned}
(\phi u)(\sqrt{-2t}, x) = &-i \int_{-\infty}^{t} \frac{1}{\sqrt{4\pi(t-\tau)}} \int \exp\left[-\frac{(x-y)^2}{4(t-\tau)}\right] \frac{g(\sqrt{-2\tau}, y)}{\sqrt{-2\tau}}\, dy\, d\tau \\
&+ \int_{-\infty}^{t} \frac{1}{\sqrt{4\pi(t-\tau)}} \int \exp\left[-\frac{(x-y)^2}{4(t-\tau)}\right] \left\{\frac{-i[(A\phi)u](\sqrt{-2\tau}, y)}{\sqrt{-2\tau}} \right. \\
&\left. + 2(D_2\phi)(D_2 u)\right\} dy\, d\tau.
\end{aligned} \tag{7}
$$

Changing back into the original coordinates (x_1, x_2) (where $x_1 > 0$) we find that

$$
(\phi u)(x_1, x_2) = v(x_1, x_2) + w(x_1, x_2) \tag{8}
$$

where

$$(9)\quad v(x_1,x_2) = -i\int_{x_1}^{\infty}\frac{1}{\sqrt{2\pi(y_1^2-x_1^2)}}\int \exp\left[\frac{-(x_2-y_2)^2}{2(y_1^2-x_1^2)}\right]g(y_1,y_2)dy_2dy_1$$

and

$$(10)\quad w(x_1,x_2) = \int_{x_1}^{\infty}\frac{1}{\sqrt{2\pi(y_1^2-x_1^2)}}\int \exp\left[\frac{-(x_2-y_2)^2}{2(y_1^2-x_1^2)}\right]\cdot$$

$$\cdot\{-i[(A\phi)u](y_1,y_2)+2y_1[(D_2\phi)(D_2u)](y_1,y_2)\}dy_2dy_1.$$

Denoting the partial Fourier transform of v with respect to the x_2 variable by $v^\wedge(x_1,\xi)$, we may rewrite (9) as

$$(11)\qquad v^\wedge(x_1,\xi) = -i\int_{x_1}^{\infty}\exp\left[\frac{-(y_1^2-x_1^2)\xi^2}{2}\right]g^\wedge(y_1,\xi)dy_1$$

Since $g = \phi f$ has a compact support, the integration in (11) is actually performed on a finite interval (of length at most equal to $\sqrt{2K}$). Moreover, for every positive number N there exists a constant C_N such that $|g^\wedge(x_1,\xi)| = |(\phi f)^\wedge(x_1,\xi)| \leqq C_N(1+|\xi|)^{-N}$ since $\phi f \in C_0^\infty(\Omega)$. Hence

$$(12)\qquad |v^\wedge(x_1,\xi)| \leqq \int_{x_1}^{\infty}|g^\wedge(y_1,\xi)|dy_1 \leqq \sqrt{2K}\,C_N(1+|\xi|)^{-N}$$

It follows that $v(x_1,x_2)$ is infinitely differentiable with respect to x_2, and each of the derivatives $D_2^k v$ is uniformly bounded as $x_1 \to 0_+$. Noting that (11) (or (9)) imply that

$$(13)\qquad D_1 v = g - ix_1 D_2^2 v$$

we see that the function $D_1 v$ along with each of its derivatives with respect to x_2 are bounded as $x_1 \to 0_+$. Differentiating (13) with respect to x_1 we find that $D_1^2 v = D_1 g - ix_1 D_2^2 v - D_2^2 v$ is uniformly bounded as $x_1 \to 0_+$, and that the same holds for each of its derivatives with respect to x_2. Iteration of this procedure leads us to the conclusion that each of the derivatives of v is uniformly bounded as $x_1 \to 0_+$ (and therefore v has in fact an infinitely differentiable extension to the *closed* half plane $\{(x_1, x_2) : x_1 \geqq 0\}$.

Turning now our attention to $w(x_1,x_2)$, we note first that dist(supp grad ϕ, V) > 0, since $\phi \equiv 1$ on the set V_1 which contains cl(V) in its interior. Moreover, the functions $A\phi$ and $D_2\phi$ have a compact support, and the fundamental solution $E(x,t)$ defined by the equations

$$E(x,t) = \begin{cases} \dfrac{1}{\sqrt{4\pi t}} e^{-x^2/t} & t > 0 \\ 0 & t \leqq 0 \text{ and } x \neq 0 \end{cases}$$

is infinitely differentiable except at the point $x = t = 0$. Hence, the functions $\psi(y_1, y_2)$ defined by the equations

$$\psi(y_1, y_2, x_1, x_2) = \begin{cases} (A\phi)(y_1, y_2)E\left(x_2 - y_2, \dfrac{y_1^2 - x_1^2}{2}\right) & y_1 \geqq 0 \\ 0 & y_1 \leqq 0 \end{cases}$$

are in fact test functions in $C_0^\infty(\Omega)$ (of the variables y_1, y_2) and depend in an infinitely differentiable manner (as vector valued functions with values in $C_0^\infty(\Omega)$ of y_1, y_2) on x_1 and x_2, where $(x_1, x_2) \in V$ and $x_1 \geqq 0$. Since u is a distribution ($u \in \mathscr{D}'(\Omega)$) and thus is continuous on $C_0^\infty(\Omega)$ it follows that the scalar function

$$\int_{x_1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(y_1^2 - x_1^2)}} \exp\left[\frac{-(x_2 - y_2)^2}{2(y_1^2 - x_1^2)}\right] [(A\phi)(y_1, y_2) \cdot u(y_1, y_2)] dy_2 dy_1$$

$$= u(E(x_2 - \cdot, \frac{\cdot^2 - x_1^2}{2})(A\phi)(\cdot))$$

is infinitely differentiable with respect to x_1 and to x_2 in the intersection of V with the closed half plane $\{(x_1, x_2) : x_1 \geqq 0\}$. Since $D_2 u$ is also a distribution in $\mathscr{D}'(\Omega)$ we may treat the second term in (10) in a similar way and conclude that $w(x_1, x_2)$ is infinitely differentiable in the closed half space $\{(x_1, x_2) : x_1 \geqq 0\}$. Using (8) we thus see that the function $u(x_1, x_2)$ and every derivative of it are uniformly bounded (in V) as $x_1 \to 0_+$. (Compare also with Corollary 3a in [1] and the remarks following it.)

In a similar fashion, let us define $v(x_1, x_2)$ and $w(x_1, x_2)$ for $x_1 < 0$ by the equations

$$(9') \quad v(x_1, x_2) = i \int_{-\infty}^{x_1} \frac{1}{\sqrt{2\pi(y_1^2 - x_1^2)}} \int \exp\left[\frac{-(x_2 - y_2)^2}{2(y_1^2 - x_1^2)}\right] g(y_1, y_2) dy_2 dy_1$$

and

$$(10') \qquad w(x_1,x_2) = \int_{-\infty}^{x_1} \frac{1}{\sqrt{2\pi(y_1^2 - x_1^2)}} \int \exp\left[\frac{-(x_2-y_2)^2}{2(y_1^2-x_1^2)}\right] \cdot \{i[(A\phi)u](y_1,y_2) - 2y_1[(D_2\phi)(D_2u)](y_1,y_2)\}dy_2dy_1$$

It follows once again that $\phi u = v + w$ for $x_1 < 0$ and that the function $u(x_1,x_2)$ as well as each of its derivatives are bounded as $x_1 \to 0_-$.

Summing up, see that we have proved, up to now, that the function $u(x_1,x_2)$ possesses C^∞ boundary values as x_1 tends to zero from either the right or the left. In order to finish the proof that $u \in C^\infty(V)$ we have to show that the boundary values of u and its derivatives actually match up and that u has no "singular part" with support on the line $x_1 = 0$. That is, we have to demonstrate that $\lim_{x_1\to 0}[(D^\alpha u)(x_1,x_2) - (D^\alpha u)(-x_1,x_2)] = 0$ for all multi-indices α, and that the distribution $v \in \mathscr{D}'(\Omega)$, defined by

$$(14) \qquad v(\phi) = u(\phi) - \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right]\int_{-\infty}^{\infty} u(x_1,x_2)\phi(x_1,x_2)dx_2dx_1$$

for $\phi \in C_0^\infty(\Omega)$, is the zero distribution. The functional $v(\phi)$ is well defined because of the existence of boundary values for $u(x_1,x_2)$ as $x_1 \to 0_+$ and $x_1 \to 0_-$, and obviously supp $v \subset \{(x_1,x_2) : x_1 = 0\}$. According to [8, pp. 100–101], we may write

$$(15) \qquad v = \sum_{j\geqq 0} Ev_j(x_2)D_1^j$$

when $v_j(x_2)$ are one dimensional distributions defined on functions of x_2, E is the natural inclusion (extension) map $E : \mathscr{D}'(R^1 \cap V) \to \mathscr{D}'(R^2 \cap V)$ (E is the adjoint of the restriction map $R : C_0^\infty(R^2 \cap V) \to C_0^\infty(R^1 \cap V)$ where $R^1 = \{(x_1,x_2) : x_1 = 0\}$), and the sum is locally finite. Then

$$(16) \qquad Av = \sum_j Ev_j(x_2)D_1^{j+1} + \sum_j ix_1E(D_2^2v_j)(x_2)D_1^j.$$

For all $\phi \in C_0^\infty(V)$ and $w \in \mathscr{D}'(V)$,

$$(ix_1wD_1^j)(\phi) = w[D_1^j(ix_1\phi)] = jw(D_1^{j-1}\phi) + w(ix_1D_1^j\phi) \text{ for } j \geqq 1.$$

Setting $w = E(D_2^2 v_j)(x_2)$, and noting that

$$E(D_2^2 v_j)(x_2)(ix_1 D_1^j \phi) = (D_2^2 v_j)(x_2)[ix_1 D_1^j \phi|_{x_1=0}] = (D_2^2 v_j)(x_2)(0) = 0,$$

we conclude that

$$Av = \sum_j Ev_j(x_2) D_1^{j+1} + \sum_{j \geqq 1} ijE(D_2^2 v_j)(x_2) D_1^{j-1} \tag{17}$$

(we also use the fact that $ix_1 E(D_2^2 v_0) D_1^0 = 0$). Moreover, setting $\lim_{x_1 \to 0} [u(x_1, x_2) - u(-x_1, x_2)] = v_{-1}(x_2)$, we find that the contribution to Au of this "jump" is equal to $iEv_{-1}(x_2) D_1^0 = iEv_{-1}(x_2)$. Since $Au = f$ and f is a continuous function at $x_1 = 0$, it follows from (17) that the v_j with the highest index j has to vanish, and therefore all the v_j with $j \geqq 0$ have to vanish, (so that $v \equiv 0$). Hence, the term $iEv_{-1}(x_2)$ cannot be compensated by Av, and therefore $\lim_{x_1 \to 0} [u(x_1, x_2) - u(-x_1, x_2)] = 0$, and it follows from the invariance of A to translations in the x_2 variable that all the derivatives of the form $D_2^k u$ are in fact continuous at $x_1 = 0$. Differentiating the equation $Au = f$ and iterating the argument, we prove that for all positive integers k and l, $\lim_{x_1 \to 0} [D_1^k D_2^l u(x_1, x_2) - (D_1^k D_2^l u)(-x_1, x_2)] = 0$. Since we have shown that the distribution v (which is defined in (14)) vanishes identically, we conclude that u is a C^∞ function in a full neighborhood of the line $x_1 = 0$.

Let us note that the function

$$u(x_1, x_2) = \sum_{n=1}^{\infty} \frac{\exp\left[inx_2 - \frac{x_1^2 n^2}{2}\right]}{n^5}$$

is a classical solution of the equation $A^* u = 0$, but u is not infinitely differentiable on the line $x_1 = 0$. Hence, $A^* = D_1 - ix_1 D_2^2$ is not hypoelliptic on that line. (Of course, the fact that A^* is not hypoelliptic follows also abstractly from the unsolvability of A).

3. Proof that A is not locally solvable on the line $x_1 = 0$.

The fact that the differential operator $A = D_1 + ix_1 D_2^2$ is not locally solvable on the line $x_1 = 0$ is intimately related (as is easily seen by considering $A[u(x_1, x_2) - u(-x_1, x_2)]$) to a certain property of Cauchy problem for the backward heat operator; namely, to the fact that there exist C^∞ functions (in the closed half plane $\{(t, x) : t \geqq 0\}$) f such that there exists no solution u of the equation $\partial u / \partial t + \partial^2 u / \partial x^2 = f$ with $u(x, 0) \equiv 0$. This latter fact is probably well known, but we know of no reference to it. Therefore, we will give here a proof of the unsolvability of A. This proof is based on an averaging technique used by Hörmander [5] in his treatment of the characteristic Cauchy problem. For our (rather special) choice of an operator A, the argument is greatly simplified.

It suffices to prove that the origin is not a solvable point for A. According to a well known theorem of Hörmander (lemma 6.1.2 of [2], compare also [6]) the origin is not a solvable point for A if for every open set Ω containing the origin there exist functions f_λ, v_λ, depending on a real parameter λ and belonging to $C_0^\infty(\Omega)$ such that

$$\lim_{\lambda\to\infty}\left|\int f_\lambda v_\lambda\,dx\right| = \infty \tag{18}$$

$$\text{for every } k, \limsup_{\lambda\to\infty} \sum_{|\alpha|\leq k} \sup_x |(D^\alpha f_\lambda)(x)| < \infty \tag{19}$$

$$\text{for every } N, \limsup_{\lambda\to\infty} \sum_{|\beta|\leq N} \sup_x |(D^\beta A^* v_\lambda)(x)| < \infty\,. \tag{20}$$

It is natural to use the solutions

$$v(x_1, x_2\,;\,\xi) = \exp\left[i\xi x_2 - \frac{\xi^2 x_1^2}{2}\right]$$

of the homogeneous equation $A^*v = 0$. We cannot apply them in a straightforward manner, however, since they are not at all small as $x_2 \to \infty$. We remedy this by averaging $v(x_1, x_2\,;\,\xi)$ with respect to a certain (suitably chosen) density. We choose the density function to be equal to $1/\sqrt{2\pi\lambda}\exp[-(\xi-\lambda)^2/2\lambda]$ and obtain the function

$$u(x_1, x_2\,;\,\lambda) = \frac{1}{\sqrt{\lambda x_1^2 + 1}}\exp\left[\frac{-\lambda^2 x_1^2 - \lambda x_2^2 + 2i\lambda x_2}{2(\lambda x_1^2 + 1)}\right] \tag{21}$$

(It is easy to verify by a direct computation that $u(x_1, x_2\,;\,\lambda)$ is indeed a solution of the equation $A^*u = 0$). Let now Ω be an arbitrary (but fixed) open set containing the origin, and let $\delta > 0$ be a fixed number such that $\{x = (x_1, x_2)\colon \|x\| < \delta\} \subset \Omega$. We may assume, of course, that $2\delta < 1$. Note that

$$\frac{\lambda^2 x_1^2 + \lambda x_2^2}{2(\lambda x_1^2 + 1)} \geq \frac{\lambda\delta^2}{2} \tag{22}$$

if either $|x_1| \geqq \delta$ and $\lambda \geqq 2$ or else, if $|x_2| \geqq \delta$. From now on, let C_1, C_2 and so forth denote constants which depend on various parameters but which will be independent of λ and δ. It is clear that for every multi-index α

$$(23)\quad (D^\alpha u)(x_1, x_2\,;\lambda) = R_\alpha(x_1, x_2, \lambda, \sqrt{\lambda x_1+1})\exp\left[\frac{-\lambda^2 x_1^2 - \lambda x_2^2 + 2i\lambda x_2}{2(\lambda x_1^2+1)}\right]$$

where R_α is a rational function of its arguments, which is regular if $\sqrt{\lambda x_1^2 + 1} \neq 0$. Hence, it follows from (22) that there exist constants $C_1(\alpha)$, $C_2(\alpha)$ such that

$$(24)\qquad |(D^\alpha u)(x_1, x_2\,;\lambda)| \leqq C_1(\alpha)\lambda^{C_2(\alpha)} \exp\left[\frac{-\lambda\delta^2}{2}\right]$$

for $\|x\| \leqq 1$, $|x_1| \geqq \delta$ and $\lambda \geqq 2$ or $|x_2| \geqq \delta$.

Let $\phi \in C_0^\infty(R^2)$ be a fixed non-negative function such that $\phi(x) \equiv 1$ for $\|x\| \leqq 1$, $\phi(x) \equiv 0$ for $\|x\| \geqq 2$. Then it follows from (24) that

$$(25)\qquad \left| D^\alpha A^* \left[\phi\left(\frac{x}{\delta}\right)u(x;\lambda)\right]\right| \leqq C_3(\alpha)\delta^{-|\alpha|-2}\lambda^{C_4(\alpha)} \exp\left[\frac{-\lambda\delta^2}{2}\right]$$

Choose now a function $F(x) \in C_0^\infty(\Omega)$ such that

$$(26)\qquad \iint F(x_1, x_2)(\exp\left[\frac{-x_1^2 + 2ix_2}{2}\right] dx_1 dx_2 = a \neq 0$$

Such a function F clearly exists. Then

$$(27)\qquad \lim_{\lambda\to\infty} \lambda \int F(\lambda x)\phi\left(\frac{x}{\delta}\right)u(x\,;\lambda)dx$$

$$= \lim_{\lambda\to\infty} \iint F(y_1, y_2)\phi\left(\frac{y}{\delta\lambda}\right)\frac{1}{\sqrt{\frac{y_1^2}{\lambda}+1}}\exp\left[\frac{-y_1^2 - \frac{y_2^2}{\lambda} + 2iy_2}{2\left(\frac{y_1^2}{\lambda}+1\right)}\right]dy_1 dy_2$$

$$= \phi(0)\iint F(y_1, y_2)\exp\left[\frac{-y_1^2 + 2iy_2}{2}\right]dy_1 dy_2 = a \neq 0$$

(We recall that $\phi(0) = 1$). Note that for fixed k, the function $f_\lambda(x) = \lambda^{-k-1}F(\lambda x)$ satisfies the inequality

$$(28)\qquad \sum_{|\alpha|\leqq k} \sup_x |(D^\alpha f_\lambda)(x)| \leqq C_5(k)\lambda^{-1}$$

Defining now $v_\lambda(x) = \lambda^{k+3}u(x\,;\lambda)\phi(x/\delta)$, we infer from (25) that (20) is satisfied for fixed N, since the exponential decrease kills all the constants, the powers of δ (which are fixed) and the powers of λ. The relations (18) and (19) follow from the relations (27) and (28), respectively.

Note that A^* is explicitly solvable; a solution of the equation $A^*u = f$ might be represented by its partial Fourier transform,

$$u^\wedge(x_1, \xi) = i \int_0^{x_1} e^{-(x_1^2 - y_1)^2\xi^2/2} \; f^\wedge(y_1, \xi)\,dy_1$$

This formula makes sense at least for all $f \in C_0^\infty(R^2)$. (Of course, an abstract proof of the solvability of A^* could be based on the fact that A is hypoelliptic).

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